Additivity questions and tensor powers of random quantum channels

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Quantum states and channels

- A quantum state $\rho$ (in finite dimension) is a positive semi-definite Hermitian operator with trace one on a Hilbert space $\mathbb{C}^n$.

- A channel can be written as

$$\Phi(\rho) = \text{Tr}_{\mathbb{C}^k} [V \rho V^*]$$

Here, $V : \mathbb{C}^l \rightarrow \mathbb{C}^k \otimes \mathbb{C}^n$ is a partial isometry. This means that a channel is completely positive and trace preserving.
Complementary channels

When the input $\rho = |x\rangle\langle x|$ is a rank-one projection the following two matrices share the same non-zero eigenvalues.

$$\text{Tr}_{\mathbb{C}^k} [V|x\rangle\langle x|V^*] \sim \text{Tr}_{\mathbb{C}^n} [V|x\rangle\langle x|V^*] \quad \left( \sim \text{diag}(r_1, \ldots, r_d) \right)$$

Indeed, $V|x\rangle \in \mathbb{C}^k \otimes \mathbb{C}^n$ has the Schmidt decomposition:

$$V|x\rangle = \sum_{i=1}^{d} \sqrt{r_i} |u_i\rangle \otimes |v_i\rangle$$

where $r_i > 0$ is a probability distribution, and $\{u_i\}$, $\{v_i\}$ are orthonormal in $\mathbb{C}^k$ and $\mathbb{C}^n$.

We define the complementary channel of $\Phi$ by $^1$

$$\Phi^c(\rho) = \text{Tr}_{\mathbb{C}^n} [V\rho V^*]$$

$^1$[Holevo][King, Matsumoto, Nathanson, Ruskai]
Minimum output entropy (MOE)
The minimal output entropy of channel $\Phi$ is defined by

$$S_{\text{min}}(\Phi) = \min_\rho S(\Phi(\rho))$$

where $\rho$ are input states. [King, Ruskai]

Here, the von Neumann entropy $S(\cdot)$ of quantum state $\rho$ is:

$$S(\rho) = - \text{Tr}[\rho \log \rho] = - \sum_{i=1}^{d} \lambda_i \log \lambda_i$$

where $\lambda_i$ are eigenvalues of $\rho$. Note that

$$S(\rho \otimes \sigma) = S(\rho) + S(\sigma)$$
Holevo capacity (HC)
Holevo capacity of channel $\Phi$ is defined as:

$$\chi(\Phi) = \max_{p_i, \sigma_i} \left[ S(\Phi(\sum_i p_i \sigma_i)) - \sum_i p_i S(\Phi(\sigma_i)) \right]$$

where $\{p_i, \sigma_i\}$ is an ensemble. [Holevo][Schumacher, Westmoreland]

We have an easy bound: $\chi(\Phi) \leq \log d - S_{\text{min}}(\Phi)$

The above bound is saturated when, for example,

$$\Phi(U_g \rho U_g^*) = U_g \Phi(\rho) U_g^*$$

where $g \mapsto U_g \cdot U_g^*$ is an irreducible representation. [Holevo]
Remarks on MOE and HC

- MOE measures purity of channels by considering optimal output while HC is connected to the capacity $C(\cdot)$:

$$C(\Phi) = \lim_{n \to \infty} \frac{1}{n} \chi(\Phi^\otimes n)$$

Without entangled inputs or if additivity of $\chi$ holds, then

$$C(\Phi) = \chi(\Phi)$$

- Since von Neumann entropy is concave, MOE is achieved by pure input states. This means,

$$S_{\text{min}}(\Phi) = S_{\text{min}}(\Phi^c)$$

- To calculate HC, we need to know about more than just one output state, and in general

$$\chi(\Phi) \neq \chi(\Phi^c)$$
Additivity violation
Write quantum channels:

$$\Phi(\rho) = \text{Tr}_{\mathbb{C}^k} [V \rho V^*]$$

and their complex conjugate channels:

$$\bar{\Phi}(\rho) = \text{Tr}_{\mathbb{C}^k} [\bar{V} \rho V^T]$$

Then, with high probability we have additivity violation $^2$:

$$S_{\min}(\Phi \otimes \bar{\Phi}) < S_{\min}(\Phi) + S_{\min}(\bar{\Phi})$$

Note that, for any channels $\Phi$ and $\Omega$,

$$\min_{\rho \otimes \sigma} S((\Phi \otimes \Omega)(\rho \otimes \sigma)) = \min_{\rho} S(\Phi(\rho)) + \min_{\sigma} S(\Omega(\sigma))$$

$^2$[Hastings]: more precisely, another model was used.
Hastings proved:

\[ S_{\text{min}}(\Phi) + S_{\text{min}}(\bar{\Phi}) - S_{\text{min}}(\Phi \otimes \bar{\Phi}) \sim \frac{\log k}{k} \]

by using a “random random unitary channel” with \(1 \ll k \ll n:\)

\[ \Phi(\rho) = \sum_{i=1}^{k} r_i U_i \rho U_i^\star \]

where

- \( U_i \in \mathcal{U}(n) \) are i. i. d.
- \( r_i \sim \frac{\sum_{j=2n(i-1)+1}^{2ni} X_j^2}{\sum_{j=1}^{2nk} X_j^2} \)

where \( X_i \) are i. i. d. normal distributions.
Entangled inputs can improve the capacity - sketchy

1. We know that there is a channel such that

\[ S_{\text{min}}(\Phi \otimes \bar{\Phi}) = S_{\text{min}}(\Phi) + S_{\text{min}}(\bar{\Phi}) - \epsilon \quad \text{where} \quad \epsilon > 0 \]

2. This implies that \( \Omega = \Phi \oplus \bar{\Phi} \) gives

\[ S_{\text{min}}(\Omega^{\otimes 2}) = 2S_{\text{min}}(\Omega) - \epsilon \]

Then, there exists a channel \( \Psi \) such that

\[ \chi(\Psi \otimes \Psi) = 2\chi(\Psi) + \epsilon \]

3. So, we have

\[ C(\Psi) = \lim_{n \to \infty} \frac{1}{2n} \cdot \chi(\Psi^{\otimes 2n}) \geq \lim_{n \to \infty} \frac{1}{2} \cdot \chi(\Psi^{\otimes 2}) = \chi(\Psi) + \frac{\epsilon}{2} \]

I.e., entangled inputs improve the classical capacity: \( C(\cdot) \).

\[ ^3\text{[Fukuda, Wolf]} \]

\[ ^4\text{[Shor]} \]
Additivity question for regularized quantities

- Classical capacity:

\[ C(\Phi \otimes \Omega) \overset{?}{=} C(\Phi) + C(\Omega) \quad \text{for } \Phi \neq \Omega \]

- Regularized minimum output entropy:

\[ \hat{S}_{\text{min}}(\Phi \otimes \Omega) \overset{?}{=} \hat{S}_{\text{min}}(\Phi) + \hat{S}_{\text{min}}(\Omega) \quad \text{for } \Phi \neq \Omega \]

Here,

\[ \hat{S}_{\text{min}}(\Phi) = \lim_{N \to \infty} \frac{1}{N} \cdot S_{\text{min}}(\Phi \otimes N) \]

Remark: Non-additivity of \( \hat{S}_{\text{min}}(\cdot) \) implies non-additivity of \( C(\cdot) \).
Finding counterexamples
Concrete counterexamples for $1 \leq p \leq 2$ are still open.

Remark:
- Concrete counterexamples for the following additivity violation were found [Grudka, M. Horodecki, Pankowski]:

\[ S_{p,\min}(\Phi \otimes \Phi) < S_{p,\min}(\Phi) + S_{p,\min}(\Phi) \quad p > 2 \]

Here,
\[ S_{p,\min}(\Phi) = \min_{\rho} S_p(\Phi(\rho)) \]

where $S_p$ is the Renyi $p$-entropy: $S_p(\sigma) = \frac{p}{1 - p} \log \|\sigma\|_p$.

- Irreducible subspaces of group representations are being investigated by Brannan and Collins.
Tensor of “conjugate pair” has rather small entropy

Suppose we have a quantum channel

$$\Phi(\rho) = \text{Tr}_{\mathbb{C}^n} [V \rho V^*]$$

where

$$V : \mathbb{C}^l \rightarrow \mathbb{C}^n \otimes \mathbb{C}^k$$

is an isometry. Then, for $|b\rangle$ a Bell state,

$$\langle b_k | [\Phi \otimes \bar{\Phi}(|b_l\rangle\langle b_l|)] |b_k\rangle \geq \frac{l}{kn}$$

This means that $\Phi \otimes \bar{\Phi}$ has an output with a large eigenvalue. Additivity violation for $1 < p \leq \infty$ was shown via this trick $^5$, and for $p = 1$ later.

$^5$[Hayden, Winter]
Tensor of conjugate pair - Example

The idea behind is:

\[ U \otimes \bar{U} |b_m\rangle = |b_m\rangle \]

for \( U \in \mathcal{U}(m) \).

For example, take a random unitary channel:

\[ \Psi(\rho) = \frac{1}{k} \sum_{i=1}^{k} U_i \rho U_i^* \]

so that

\[ \Psi \otimes \bar{\Psi} (|b\rangle\langle b|) = \frac{1}{k} |b\rangle\langle b| + \frac{1}{k^2} \sum_{i \neq j} (U_i \otimes \bar{U}_j) |b\rangle\langle b| (U_i^* \otimes U_j^T) \]
Single channel has rather large entropy
What are typical outputs for randomly selected channels like?

$$|a⟩⟨a| \mapsto V|a⟩⟨a|V^* = |w⟩⟨w| \mapsto \text{Tr}_{\mathbb{C}^n} [|w⟩⟨w|] = WW^*$$

- $|a⟩$ is a fixed vector in $\mathbb{C}^l$.
- $V|a⟩$ is a random vector in $\mathbb{C}^k \otimes \mathbb{C}^n$.
- $WW^*$ is the normalized Wishart matrix.

The probability density of $WW^*$ is proportional to:

$$\delta \left( 1 - \sum_{1\leq i \leq k} p_i \right) \prod_{1\leq i < j \leq k} (p_i - p_j)^2 \prod_{1\leq i \leq k} p_i^{n-k}$$

The last factor shows that $n \gg k$ implies concentration of eigenvalues. So, typical outputs have rather large entropy.
Aubrun-Szarek-Werner approach for $p > 1$

Define a random quantum channel $\Phi$ by the random isometry:

$$V : \mathbb{C}^{n^{1+1/p}} \to \mathbb{C}^n \otimes \mathbb{C}^n.$$ 

Based on the previous argument, $\Phi \otimes \bar{\Phi}$ has a large output eigenvalue larger than $n^{-1+1/p}$.

By using Dvoretzky’s theorem,

$$S_{p,\min}(\Phi) \sim S_{p,\min}(\Phi \otimes \bar{\Phi}).$$

Of course then we have violation for large $n$.

$$S_{p,\min}(\Phi) + S_{p,\min}(\bar{\Phi}) > S_{p,\min}(\Phi \otimes \bar{\Phi}).$$

They later showed additivity violation for $p = 1$ by a similar technique.
What are candidates for optimal inputs for $\Phi \otimes \bar{\Phi}$?\textsuperscript{6}

Take a random quantum channels defined by

$$\Phi_n(\rho) = \text{Tr}_{C^n} [V \rho V^*]$$

with

$$V : C^l \to C^{kn}$$

where $l = t kn$, $k \in \mathbb{N}$, $t \in (0,1)$ are fixed and $n \to \infty$.

Then, we investigated the asymptotic behavior (as $n \to \infty$) of output eigenvalues of

$$Z_n = \Phi_n \otimes \bar{\Phi}_n(|a_n\rangle\langle a_n|)$$

where $(a_n)_{n \in \mathbb{N}}$ is a fixed sequence of unit vectors.

\textsuperscript{6}[Collins, F, Nechita]
We found that the empirical eigenvalue distribution of the matrix $Z_n$ converges \textit{almost surely}, as $n \to \infty$, to:

$$
\frac{1}{k^2} \left[ \delta_{\lambda_1} + (k^2 - 1)\delta_{\lambda_2} \right] \, dx
$$

where the Dirac masses are located at

$$
\lambda_1 = t|m|^2 + \frac{1 - t|m|^2}{k^2} \quad \text{and} \quad \lambda_2 = \frac{1 - t|m|^2}{k^2}.
$$

if

$$
\frac{\text{Tr} [A_n]}{\sqrt{l}} = m + O \left( \frac{1}{n^2} \right)
$$

Here, $|a_n\rangle \leftrightarrow A_n$ is the correspondence $\mathbb{C}^l \otimes \mathbb{C}^l \leftrightarrow M_l(\mathbb{C})$.

Conclusion: The Bell state is best. Examine, for example,

$$
a = \sum_i \alpha_i |i\rangle \otimes |i\rangle
$$

Remark. Nothing interesting with $\Phi \otimes \Phi$, $\Phi \otimes \Phi^T$ or $\Phi \otimes \Phi^*$. 
How about tensor powers \((\Phi \otimes \bar{\Phi})^\otimes r\)?

Our calculation shows that tensor-products of Bell states are best. Suppose we have a random quantum channel:

\[
\Phi \otimes \Phi \otimes \ldots \otimes \Phi \otimes \hat{\Phi} \otimes \hat{\Phi} \otimes \ldots \otimes \hat{\Phi}
\]

where best inputs are

\[
|b_{\pi(1),\hat{1}}\rangle \otimes |b_{\pi(2),\hat{2}}\rangle \otimes \cdots \otimes |b_{\pi(r),\hat{r}}\rangle
\]

where \(\pi \in S_r\). Here, \(|b_{i,j}\rangle\) is a Bell state over the \(i\)-th space for \(\Phi\) and \(j\)-th space for \(\bar{\Phi}\).

**Remark.** Hastings conjectured that violation of additivity happens only within each conjugate pair.
How about tensor powers $\Phi \otimes^{2r}$, where $\Phi$ is orthogonal?\footnote{[F, Nechita]}  
This time, we generate random channels by orthogonal matrices instead of unitary ones. So, $\bar{\Phi} = \Phi$. 

$$
\phi_1 \otimes \phi_2 \otimes \ldots \otimes \phi_r \otimes \phi_{r+1} \otimes \phi_{r+2} \otimes \ldots \otimes \phi_{2r}
$$

where best inputs are 

$$
\bigotimes_{c \in \pi} |b_c\rangle
$$

where $\pi$ is a paring of $2r$ elements. Here, $|b_c\rangle$ is a Bell state over the $i$-th and $j$-th spaces when $c = (i, j)$.

We conjecture that typically for orthogonal case

$$
S_{\min}(\Phi \otimes^{2r}) = r S_{\min}(\Phi \otimes^2)
$$
or, we can make it weaker:

$$
\lim_{r \to \infty} \frac{1}{r} S_{\min}(\Phi \otimes^r) = \frac{1}{2} S_{\min}(\Phi \otimes^2)
$$
Montanaro’s multiplicative bound

$$\|\Phi^\otimes r\|_{1\rightarrow \infty} \leq \left(\| (V V^*)^\Gamma \|_\infty \right)^r$$

where $V$ is the isometry defining $\Phi$.

F-Nechita’s multiplicative bound

$$\|\Phi^\otimes r\|_{1\rightarrow 2} \leq \left(\| C_\Phi^\Gamma \|_\infty \right)^r$$

where $C_\Phi^\Gamma$ is the partially transposed Choi matrix of $\Phi$.

Then the bounds lead to the following weak additivity respectively for $p = \infty, 2$: typically under random choice of channels

$$S_{p,\min}(\Phi^\otimes r) \geq \frac{r}{2} S_{p,\min}(\Phi)$$

Montanaro first described it as “weakly multiplicative”, in terms of maximum output $p$-norms.
F-Gour’s multiplicative bound (no random here)
For a unital quantum channel: $M_n(\mathbb{C}) \rightarrow M_k(\mathbb{C})$, 

$$ \| \Phi \otimes r \|_{1 \rightarrow 2} \leq (\gamma \Phi)^{r/2} . $$

Here,

$$ \gamma \Phi = \frac{1}{k} + \left(1 - \frac{1}{n}\right) \| D\Phi D\Phi^* \|_\infty $$

where $D\Phi$ is the dynamical matrix of $\Phi$ restricted on trace-less Hermitian matrices.

We also got an upper bound for the classical capacity:

$$ C(\Phi) \leq \log k + \log \gamma \Phi . $$

This bound is saturated by the Werner-Holevo channel.
Summary

- Additivity violation may be a special phenomena only for conjugate pairs.
- Perhaps, additivity violation typically does not hold for $\Phi^\otimes n$ when $\Phi$ is generated by unitary group.
- Otherwise, we need to know how fast non-additivity grows and how much contribution it makes for regularized quantity.

Thank you very much.

Acknowledgement: JSPS KAKENHI Grant Number JP16K00005